

General Measurements with Limited Resources and Their Application to Quantum Unambiguous State Discrimination

Jan Bouda

Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic

Daniel Reitzner

Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic and RCQI, Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia

In this report we present a framework for implementing arbitrary n -outcome quantum measurement as a sequence of two-outcome measurements requiring only single ancillary qubit. Our procedure is the same as one presented in [1] but in addition offers particular construction for a two-outcome partial measurements. We exemplify this framework on the unambiguous state discrimination. In the simplest case it gives the same construction as is known, if we opt for performing conclusiveness measurement first. However, it also offers possibility of performing measurement for one of the state outcomes, which shows flexibility of presented framework.

I. INTRODUCTION

Measurements are one of the crucial elements of quantum theory. Compared to the usual notions of observables as self-adjoint operators, or to von Neumann measurements, there exist more general measurements, corresponding to the so-called positive operator-valued measures (POVMs) defined by a set of positive operators summing up to the identity operator [2]. Being more general, POVMs outperform projective measurements for many tasks in quantum information theory, including quantum tomography [3], unambiguous discrimination of quantum states [4], state estimation [5], quantum cryptography [6–8], information acquisition from a quantum source [9], Bell inequalities [10, 11] or device-independent quantum information protocols [12, 13].

Near term quantum computation devices at present provide only limited number of resources. Firstly, their size is limited only to handful of qubits. Secondly, number of operations and their precision is still very limited by the demands on the technology. Finally, a more technical detail are only limited possibilities of measurements — current devices perform only noisy versions of projective measurements in computational basis.

Our aim is to provide some possibilities of performing POVM measurements with these limited resources with focus mostly on the spatial-temporal compromises and not on the precision of measurements. Some thought on the latter can be found in [14, 15],

Suppose we want to perform an n -outcome measurement in d -dimensional Hilbert space, $\mathbf{A} = \{A_j\}_{j=1}^n$. By naive interpretation of Naimark one can need ancillary Hilbert space of dimension up to dn . This is a single measurement procedure, where the measurement on the whole space at some point provides whole outcome information. On the other side lies the result of Ozmaniec [16] where the spatial resources are exchanged for temporal resources. The whole measurement is performed on the system Hilbert space and is successful only with probability $1/d$.

On one hand simple dilation may need more resources than are at hand, on the other hand measurements just on given Hilbert space are just probabilistic and one does not have a direct access to the measurement \mathbf{A} . In this case one can only reconstruct statistics by post-selecting obtained data. We would like to explore the possible trade-off between the two extremes.

Inspired by [17], in this paper we will concentrate on the next simplest model to the single qubit approach, in which having single ancillary qubit that in ideal scenario shall provide us with possibility of performing simple (two-outcome) measurements. We shall explore this option from a point of view determining practical ways of performing complex measurements as a sequence of simple ones as depicted in Fig. 1.

A. Generalized measurements

Generalized measurements, or *Positive operator valued measures (POVMs)*, are a general way of describing measurements in quantum theory. In finite-outcome case we are about to study an n -outcome POVM \mathbf{A} is represented as a set of operators $\mathbf{A} = \{A_j\}_{j \in [n]}$, where $[n] = \{1, 2, \dots, n\}$. Operator A_j correspond to the outcome j ; having state ρ on which we perform measurement \mathbf{A} , outcome j is obtained with probability p_j that is given by *Born formula*, $p_j = \text{tr}[A_j \rho]$. This demands that the operator A_j is positive semi-definite, $A_j \geq 0$; these operators are called *effects*.

We also require that the probabilities sum up to one,

$$1 = \sum_{j=1}^n p_j = \sum_{j=1}^n \text{tr}[A_j \rho] = \text{tr} \left[\rho \sum_{j=1}^n A_j \right].$$

As this has to hold for all states ρ it follows that the sum of the POVM effects equals to identity,

$$\sum_{j=1}^n A_j = \mathbb{1}.$$

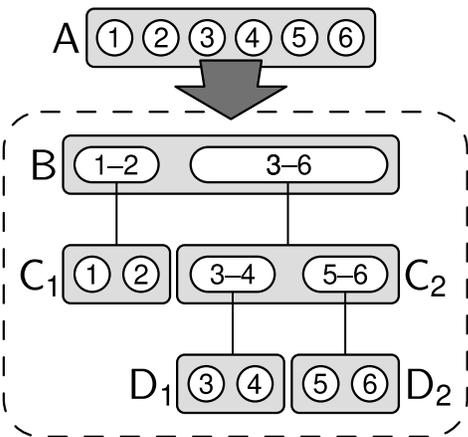


FIG. 1: Example of a coarse-graining. Imagine measurement A with six outcomes. Measurement B is a coarse-graining of A having two outcomes, one being collective outcome for outcomes 1 and 2 of measurement A and the other one being collective outcome for outcomes 3–6 of measurement A. The idea of the paper is to use two-outcome coarse-grainings used in sequential way to perform measurement A. In this case, if measurement B gives outcome 1–2, it is followed by measurement C₁ giving definitive answer ① or ②. If, however, the B-measurement gives outcome 3–6, it is followed first by measurement C₂ and based on its outcome either D₁ or D₂ is performed.

Note, that von Neumann (projective) measurements are a subset of the set of POVMs, as any projective measurement is described by a set of particular projections, which are also effects.

For the purposes of this paper we also define the notion of *coarse-graining*. Let us have a partition P of the results of the measurement $A = \{A_j\}_{j \in [n]}$, i.e. $P_k \subseteq [n]$ such that $\cup_k P_k = [n]$ and $P_j \cap P_k = \emptyset$ for all $j \neq k$. A coarse-graining is such a measurement $B = \{B_k\}_k$ that composes outcomes according to given partitioning P , $B_k = \sum_{j \in P_k} A_j$. We will also use term *fine-graining*, which is an opposite to coarse-graining, i.e. it corresponds to splitting of effects B_k to sub-effects, providing finer measurement. For example, having a measurement $A = \{A_1, A_2, A_3, A_4\}$ a coarse graining can be a measurement $B = \{B_1, B_2\}$, where $B_1 = A_1 + A_2 + A_4$ and $B_2 = A_3$. Measurement A is then a fine-graining of B.

B. Naimark dilation theorem

To provide some background, we present in this subsection a mathematical model of performing a measurement by extending given system to higher dimension where an experimentally accessible von Neumann measurements are available.

Let $\{F_i\}_{i=1}^n$ be a POVM acting on Hilbert space \mathcal{H}_A of dimension d_A . Then there exists a projective measure-

ment $\{P_i\}_{i=1}^n$ acting on the Hilbert space $\mathcal{H}_{A'}$ of dimension $d_{A'}$ and an isometry $S : \mathcal{H}_A \rightarrow \mathcal{H}_{A'}$ such that for all i

$$F_i = V^\dagger P_i V. \quad (1)$$

A naive (and inefficient) way to construct such projective measurement and isometry is to let $\mathcal{H}_{A'} = \mathcal{H}_A \otimes \mathcal{H}_B$, $P_i = I_A \otimes |i\rangle_B \langle i|$, and

$$V = \sum_{i=1}^n \sqrt{F_i} \otimes |i\rangle_B. \quad (2)$$

This construction, however, requires system of large dimension, and $d_{A'} = nd_A$. This approach of POVM can be turned into physical implementation by extending the isometry S to a unitary operation U given by

$$S = U(I_A \otimes |0\rangle_B). \quad (3)$$

More dimension-efficient approach was designed by Peres [2], where the construction requires dimension

$$d_{A'} = \sum_{i=1}^n \text{rank } F_i.$$

In this work we will similarly extend the studied system but only by a qubit system. This dilation to a qubit, however, limits possibilities for our intended measurements. Namely, we cannot expect to be able to perform a measurement with more than two outcomes (on the qubit system). This, in turn, defines a way, how we will approach the problem of measuring more outcomes — we will look at the possibility of splitting the measurements into a sequence of two-outcome measurements.

C. Measurements with state changes

POVMs describe measurements only from the perspective of outcomes and their probabilities. They do not, however, describe what happens to the measured state in any way. In the case of von Neumann measurements, the change to the state ρ when outcome j corresponding to projector P_j is measured, is given as $\tilde{\rho}_j = P_j \rho P_j$. The operator $\tilde{\rho}_j$ is unnormalized, and its normalization provides both the outcome state

$$\rho_j = \frac{P_j \rho P_j}{\text{tr}[P_j \rho]}$$

and the probability of getting this outcome, $p_j = \text{tr}[P_j \rho]$.

For general POVM, we will describe these measurement-induced state changes as *instruments*. An instrument \mathcal{I} , corresponding to measurement A is a set of completely positive trace non-increasing maps $\mathcal{I} = \{\mathcal{I}_j\}_j$ such that

$$\text{tr}[\mathcal{I}_j(\rho)] = \text{tr}[A_j \rho] \quad (4)$$

that has to hold for all states ρ . The positivity of A_j translates to the requirement that \mathcal{I}_j is completely positive, while the summation condition for A translates to the requirement that the sum of \mathcal{I}_j 's is a channel (completely positive trace preserving map) which implies the trace non-increasing property on the particular \mathcal{I}_j 's. As before, operators $\tilde{\rho}_j = \mathcal{I}_j(\rho)$, representing what happens to state ρ when outcome j is observed, are not normalized, with probability p_j of obtaining the outcome being the normalization factor, i.e.

$$\rho_j = \frac{1}{p_j} \mathcal{I}_j(\rho) = \frac{\mathcal{I}_j(\rho)}{\text{tr}[\mathcal{I}_j(\rho)]} = \frac{\mathcal{I}_j(\rho)}{\text{tr}[A_j \rho]}.$$

Important thing to note is that while for von Neumann measurements the presented state change is the only possibility, in the general case of POVMs, the choice is not unique. Different choices can affect the state in different ways and, in particular, can lead to various degrees of state disturbances. It is therefore natural to try to find the least disturbing choices, especially, when the resulting state is to be used later. It is observed [18, 19] that the so-called Lüder's measurements (or instruments), given by prescription

$$\mathcal{L}_j(\rho) = A_j^{1/2} \rho A_j^{1/2},$$

are the least disturbing in many cases — any measurement can be realized as a Lüder's measurement followed by some measurement-independent state change — and, so, we will turn to them in this study.

II. MEASURING WITH LIMITED RESOURCES

As we noted before, current quantum devices provide us with only limited resources. If a desired measurement is more complicated, these resources might not allow us to implement them. A straightforward idea then is to split the measurement into a sequence of partial ones as depicted in Fig. 1. While in classical world such action bears no problems, in quantum case we know, that every measurement disturbs measured state. A question arises now, whether it is possible to devise such procedure that would allow us to perform measurement in this sequential way. If such a procedure exists, additional questions are, whether this procedure is simply implementable and what are its limitations.

The answer to these questions has been in general provided in [1]. In this paper we present a slightly different way of obtaining the result and later, we apply this procedure to study unambiguous state discrimination. We obtain the result by first showing that the Lüder's measurements allow for fine-graining of results. Then we show how to perform these measurements for qubit measurements, which in turn allows us to implement this procedure on current quantum devices based on qubit registers.

Let us consider measurement A and its coarse-graining B . We will consider only a two-outcome coarse-graining as (i) we want to study the possibilities of single ancillary qubit that distinguishes only two outcomes, and (ii) the extension of the procedure that will be presented is straightforward. So let us have $B = \{B, \mathbb{1} - B\}$ and Q being the set of outcome indices of measurement A that define B , i.e. $B = \sum_{j \in Q} A_j$.

Let us consider now that the effect B was measured on the input state ρ . The (unnormalized) state now is $\tilde{\rho} = B^{1/2} \rho B^{1/2}$. If we now want to fine-grain the results to obtain information about outcomes from Q , we cannot perform measurement A on the state $\tilde{\rho}$ any more — the measurement to be performed needs to be adjusted for the fact that previous measurement B has been already done.

In fact, what we want is to find such measurement $A' = \{A'_j\}_{j \in Q}$ that the following holds

$$\text{tr}[\tilde{\rho} A'_j] = \text{tr}[\rho A_j]. \quad (5)$$

Expanding the left side we also see

$$\text{tr}[\tilde{\rho} A'_j] = \text{tr}[B^{1/2} \rho B^{1/2} A'_j] = \text{tr}[\rho B^{1/2} A'_j B^{1/2}].$$

Since this has to be equal to $\text{tr}[\rho A_j]$ for all ρ , we obtain condition for A'_j stating that $A_j = B^{1/2} A'_j B^{1/2}$. We can now take the pseudoinverse of $B^{1/2}$, which we denote simply as $B^{-1/2}$, and we finally obtain

$$A'_j = B^{-1/2} A_j B^{-1/2}. \quad (6)$$

Since $B \geq 0$ we also have $A'_j \geq 0$. For the summation rule

$$\sum_{j \in Q} A'_j = \sum_{j \in Q} B^{-1/2} A_j B^{-1/2} = B^{-1/2} B B^{-1/2} = \mathbb{1}_B,$$

where $\mathbb{1}_B$ is the identity on the $\text{ran } B$. We need not be concerned with the rest of the Hilbert space, as $\text{ran } A_j \subseteq \text{ran } B$ and also $\text{ran } \tilde{\rho} \subseteq \text{ran } B$. This means that while the transformed state loses information outside the $\text{ran } B$, the subsequent measurements anyway act within $\text{ran } B$. So we see that Eq. (6) is sufficient to define the subsequent measurement that fulfills the condition from Eq. (5) and the Lüder's measurement of a coarse-grained measurement retains information for obtaining fine-grained outcomes.

Without the need for particular implementation of Lüder's measurements, we can consider various possibilities of performing complex measurements in a sequential way as depicted in Fig. 1. Numerous possibilities can be considered, and the role in their usefulness might have also implementation details, noise considerations, or particularities of tasks the measurement answers. Without these considerations, two of these procedures stand out.

a. Outcome-decreasing procedure in every steps tries to rule out one of the outcomes. Having measurement $A = \{A_j\}_{j \in [n]}$, in step j we perform measurement

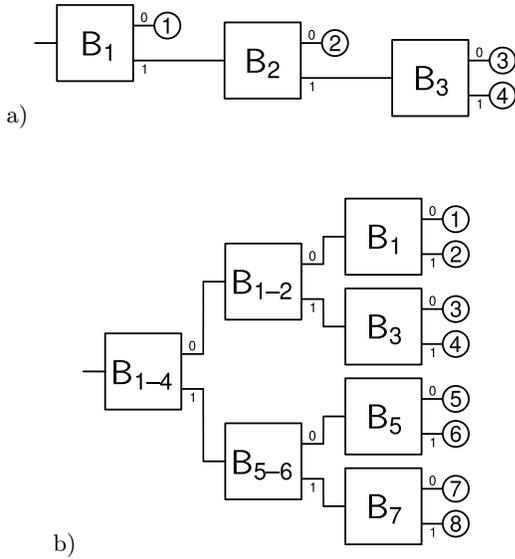


FIG. 2: Examples of possible measurement procedures. Figure a) depicts an outcome-decreasing procedure, where every measurement eliminates one outcome, in this case representing overall measurement $A = \{A_1, A_2, A_3, A_4\}$. Figure b) shows a binary-search procedure in which the number of possible outcomes is halved in every step with depicted scenario representing an eight-outcome measurement A . In both cases B_x determines measurement in which x defines the set of indices we query, with 0 standing for successful query and 1 for unsuccessful. Circled numbers represent definitive outcomes.

$B_j = \{A_j, \mathbb{1} - A_j\}$ with outcome \textcircled{j} for getting definitive answer, or outcome $\textcircled{j'}$ which rules out outcome \textcircled{j} . If outcome \textcircled{j} is obtained, the measurement process can be terminated, as a definitive answer is obtained. The procedure is depicted in Fig. 2a. This procedure is simple to implement and does not require conditioning on previous outcomes — if definitive answer is obtain, the measurement procedure can continue, but we can disregard the results. The drawback of this procedure is its time ineffectiveness as one needs to perform n steps of the measurement process.

b. Binary-search procedure splits the outcomes of current measurement in half and based on given outcome it chooses the next measurement to be done. This procedure is depicted in Fig. 2b. Compared to the outcome-decreasing procedure, it is more time efficient, as the number of steps one needs to make is roughly $\log_2 n$. The price to pay is that one needs to be able to condition measurements to be done on previous outcomes. This condition is not yet available on current quantum devices.

A. Qubit implementation of Lüder's measurements

Remaining question now is, whether there is a simple and efficient realization. In this part we shall show that using one ancillary qubit, we can represent any two-

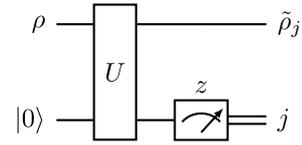


FIG. 3: General coupling scheme for simple measurements. The (not necessarily qubit) state ρ is coupled by unitary U to ancillary qubit system prepared in state $|0\rangle$, which is measured in z basis afterwards. Given outcome of the measurement is \textcircled{j} , the output state is $\tilde{\rho}_j$.

outcome measurement $B = \{B, \mathbb{1} - B\}$ as a rotation of original system to the basis of B , followed by a controlled unitary with the ancillary qubit as the target, and finalized by measurement of ancillary qubit and rotation of the system back.

The reasoning is following. Let us start with a general coupling construction as in Fig. 3, where the original state is coupled to an ancillary state which is measured afterwards. A reasonable simplifications (both mathematically and technically) are that the ancillary qubit is prepared in state $|0\rangle$ and the its measurement is in standard (z) basis. This setup, if substituted to Eq. (4), requires following conditions to hold:

$$B^{1/2} \rho B^{1/2} = {}_2\langle 0|U|0\rangle_2 \rho {}_2\langle 0|U^\dagger|0\rangle_2$$

$$(\mathbb{1} - B)^{1/2} \rho (\mathbb{1} - B)^{1/2} = {}_2\langle 1|U|0\rangle_2 \rho {}_2\langle 0|U^\dagger|1\rangle_2,$$

where we explicitly used indices marking the original system (1) and the ancillary qubit (2) where necessary. Requiring this to hold for all ρ , we have conditions for U :

$${}_2\langle 0|U|0\rangle_2 = B^{1/2} \quad \text{and} \quad {}_2\langle 1|U|0\rangle_2 = (\mathbb{1} - B)^{1/2}.$$

Since B is an effect, it can be diagonalized by a unitary transformation, let us say U_B . This unitary diagonalizes at the same time both $B^{1/2}$ and $(\mathbb{1} - B)^{1/2}$. By emphasizing the diagonal form by \mathcal{D} , we have

$$\mathcal{D}(B^{1/2}) = U_B B^{1/2} U_B^\dagger = U_{B2} {}_2\langle 0|U|0\rangle_2 U_B^\dagger$$

$$= {}_2\langle 0|(U_B \otimes \mathbb{1})U(U_B^\dagger \otimes \mathbb{1})|0\rangle_2,$$

$$\mathcal{D}((\mathbb{1} - B)^{1/2}) = U_B (\mathbb{1} - B)^{1/2} U_B^\dagger = U_{B2} {}_2\langle 1|U|0\rangle_2 U_B^\dagger$$

$$= {}_2\langle 1|(U_B \otimes \mathbb{1})U(U_B^\dagger \otimes \mathbb{1})|0\rangle_2.$$

Denoting by $V = (U_B \otimes \mathbb{1})U(U_B^\dagger \otimes \mathbb{1})$, which is unitary as well, we can write it as

$$V = \begin{pmatrix} \mathcal{D}(B^{1/2}) & \bullet \\ \mathcal{D}((\mathbb{1} - B)^{1/2}) & \bullet \end{pmatrix}. \quad (7)$$

We denote elements to be filled by \bullet . Moreover, this matrix has this block form in reverse tensor product of the system and the ancillary qubit. Denoting columns of V as v_k , it is easy to see that for $j \neq k$ in the known part we

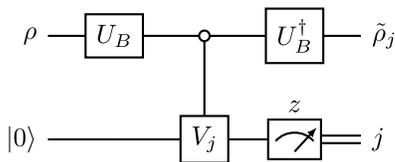


FIG. 4: The measurement scheme can be simplified by rotating ρ to basis of B by U_B (and back at the end). Then, the coupling unitary is a general control operation of form $V = \sum_j |j\rangle\langle j| \otimes V_j$.

have $v_j^* v_k = 0$ as the filled in matrices are diagonal. Denoting the eigenvalues of B as $\lambda_j \in [0; 1]$, for the column norm we have

$$\begin{aligned} v_j^* v_j &= \left(\lambda_j^{1/2}\right)^* \left(\lambda_j^{1/2}\right) + \left[(1 - \lambda_j)^{1/2}\right]^* (1 - \lambda_j)^{1/2} \\ &= \lambda_j + (1 - \lambda_j) = 1. \end{aligned}$$

So we see, that the left part of V fulfills the conditions for unitarity and, hence, we can complete V to unitary in such a way, that also completed sub-matrices are diagonal. If we now write the matrix V in the normal tensor order, it has a block-diagonal structure which is easily interpreted as a controlled operation of the form

$$V = \sum_j |j\rangle\langle j| \otimes V_j$$

for some V_j .

To sum up, the procedure from Fig. 3 can in particular be constructed as in Fig. 4, in which we first rotate the original state to B -basis, then perform controlled operation with ancillary qubit as target, and finally measure the ancillary qubit and rotate the original system back from B -basis.

Presented construction is general in the dimension of original system. In the next part we will consider the simplest case, when the original system is a qubit. In this case we can rewrite this controlled operation as

$$\begin{aligned} V &= |0\rangle\langle 0| \otimes V_0 + |1\rangle\langle 1| \otimes V_1 \\ &= (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes V_1 V_0^\dagger) (\mathbb{1} \otimes V_0), \end{aligned} \quad (8)$$

which is a composition of unitary transformation V_0 on the ancillary qubit and standard qubit controlled- $(V_1 V_0^\dagger)$ operation (see Fig. 5). For the following computations for unambiguous state discrimination we have chosen a particular completion of V details for which can be found in A.

III. QUANTUM UNAMBIGUOUS STATE DISCRIMINATION AS A SEQUENTIAL MEASUREMENT PROCESS

A. Quantum unambiguous state discrimination

Quantum state discrimination is a task when we are provided a state from a set of states $\{\rho_j\}_{j \in [m]}$ with prob-

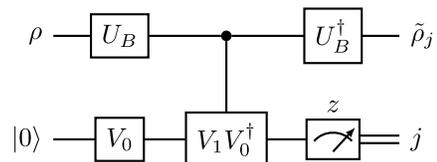


FIG. 5: The measurement scheme can be simplified even further in the case of qubit system when the general controlled operation can be constructed as a composition of V_0 on the ancilla followed by standard controlled- $(V_1 V_0^\dagger)$ operation.

abilities $\{p_j\}_{j \in [m]}$ and our task is to determine, which state was presented to us. Due to particularities of quantum mechanics, this task is not as straightforward as in the classical case — in quantum theory one cannot distinguish non-orthogonal states perfectly. This task is therefore of high importance.

A particular situation of *unambiguous state discrimination* was introduced in [20, 21]. In this setting we want to distinguish particular states without errors, i.e. if we are given a definite answer about the state, it needs to be correct. The price to pay for this requirement is the necessity for an *inconclusive* outcome. When we obtain this result, we cannot say anything about presented state.

This particular task has many extensions, but for the sake of exemplifying the framework from previous section we will deal with the most basic setting of being presented with two pure qubit states $\{|\psi_1\rangle, |\psi_2\rangle\}$ with equal probabilities. Our task is to determine, which state was given to us.

Let us first denote $|\psi_j^\perp\rangle$ as perpendicular states to $|\psi_j\rangle$ and P_j, P_j^\perp as their corresponding projectors. The unambiguous state discrimination measurement A has three outcomes $\textcircled{1}$, $\textcircled{2}$, and $\textcircled{?}$ with corresponding effects,

$$A_1 = \lambda P_2^\perp, \quad A_2 = \lambda P_1^\perp, \quad A_? = \mathbb{1} - A_1 - A_2.$$

The choice for effects A_1 and A_2 is logical, as it tells us that the presented state is not the other one. Effect $A_?$ corresponds to the inconclusive answer $\textcircled{?}$ and $\lambda \in [0; 1]$ is such a parameter that $A_? \geq 0$.

In order to analyze this situation, let us parametrize the problem (see also Fig. 6). In qubit case we can always pre-process presented states so that they would be easily described as

$$\begin{aligned} |\psi_1\rangle &= \cos \omega |0\rangle + \sin \omega |1\rangle, \\ |\psi_2\rangle &= \cos \omega |0\rangle - \sin \omega |1\rangle, \end{aligned}$$

with $\omega \in [0; \pi/4]$. The case of $\omega = 0$ corresponds to $|\psi_1\rangle = |\psi_2\rangle$, while $\omega = \pi/4$ describes orthogonal states. We will disregard this necessity for the pre-processing as it is not relevant for this work.

Within this parametrization we have

$$\begin{aligned} |\psi_1^\perp\rangle &= \sin \omega |0\rangle - \cos \omega |1\rangle, \\ |\psi_2^\perp\rangle &= \sin \omega |0\rangle + \cos \omega |1\rangle. \end{aligned}$$

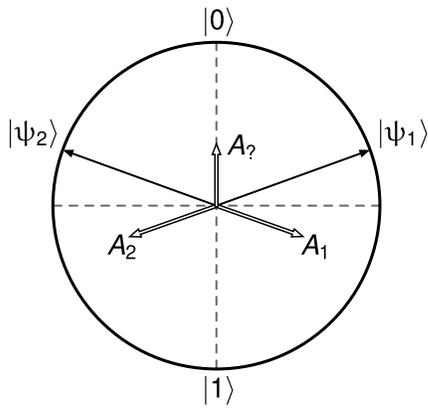


FIG. 6: Depiction of unambiguous state discrimination in the Bloch picture.

By minimizing the probability for the inconclusive outcome $\textcircled{?}$ we find the Optimal choice of λ to be

$$\lambda = \frac{1}{2 \cos^2 \omega}.$$

We can now explicitly express

$$A_{1,2} = \frac{1}{2} \begin{pmatrix} \tan^2 \omega & \pm \tan \omega \\ \pm \tan \omega & 1 \end{pmatrix}, \quad A_? = \begin{pmatrix} 1 - \tan^2 \omega & 0 \\ 0 & 0 \end{pmatrix}$$

By construction we see that $A_{1,2}$ are multiples of projector and we can also observe that $A_?$ is a multiple of projector, a measurement in the z direction.

An important quantity for us is the probability of inconclusive result,

$$p_? = \frac{1}{2} \text{tr}[E_? P_1] + \frac{1}{2} \text{tr}[E_? P_2] = \cos 2\omega.$$

The probability of conclusive result is

$$p_! = 1 - p_? = 1 - \cos 2\omega = 2 \sin^2 \omega.$$

Before analyzing particular sequential measurement scenarios, let us set the notation a bit. In the first case we will consider first the measurement $\mathbf{B} = \{A_?, \mathbb{1} - A_?\}$, where we will denote corresponding outcomes as $\textcircled{?}$ for inconclusive answer and $\textcircled{!}$ for conclusive answer. We shall call this measurement *conclusiveness measurement* as it tells us, whether in the subsequent measurement we will obtain a conclusive result or not. In the second case we will start with measurement $\mathbf{B} = \{A_1, \mathbb{1} - A_1\}$, with corresponding outcomes $\textcircled{1}$ and $\textcircled{1'}$. We will call this measurement *state 1 measurement* (or for brevity just state measurement) that either tells us whether we were given state $|\psi_1\rangle$ or whether we should continue with the measurement with possibility of obtaining result $\textcircled{2}$.

B. Conclusiveness measurement as the first measurement

Let us look first at the symmetric case in which we perform conclusiveness measurement first (see Fig. 7a) and

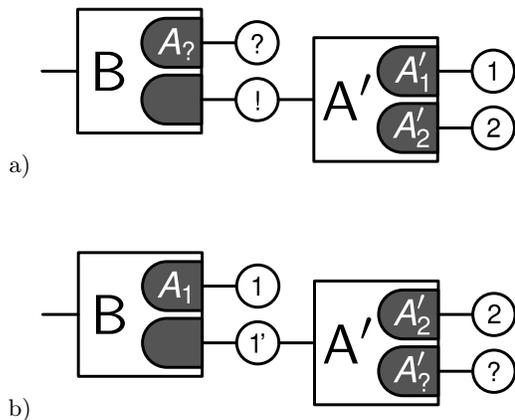


FIG. 7: Two analyzed scenarios of unambiguous state discrimination, a) presents the usual setting where conclusive measurement is performed first, while b) presents situation in which we first ask whether we are given state $|\psi_1\rangle$ and only in the case of negative outcome $\textcircled{1'}$ we perform second measurement questioning whether the state is $|\psi_2\rangle$ or we have an inconclusive answer $\textcircled{?}$.

then perform the outcome measurement. In this case we coarse-grain the measurement $\mathbf{A} = \{A_1, A_2, A_?\}$ by $\mathbf{B} = \{A_?, \mathbb{1} - A_?\}$. In this case the unitaries used in the construction of the coupling in Eq. (8) and the basis transformation U_B are determined to be

$$U_B = \mathbb{1}, \quad V_0 = \sigma_z, \\ V_1 = \begin{pmatrix} \tan \omega & -\sqrt{1 - \tan^2 \omega} \\ \sqrt{1 - \tan^2 \omega} & \tan \omega \end{pmatrix}.$$

The pre-measurement state after the coupling transformation is

$$|\psi'_{1,2}\rangle = -\sqrt{2} \sin \omega |\pm\rangle \otimes |0\rangle + \sqrt{\cos 2\omega} |1\rangle \otimes |1\rangle. \quad (9)$$

The measurement measures state $|1\rangle$ corresponding to inconclusive outcome $\textcircled{?}$ with probability $p_? = \cos 2\omega$. The (normalized) post-measurement state is the same for both initial states, $|1\rangle$.

The conclusive result $\textcircled{!}$ is obtained with probability $p_! = 2 \sin^2 \omega$ by measuring qubit in state $|0\rangle$. The (normalized) post-measurement states are $|\tilde{\psi}_1\rangle = |+\rangle$ for initial state $|\psi_1\rangle$ and $|\tilde{\psi}_2\rangle = |-\rangle$ for initial state $|\psi_2\rangle$. These states are orthogonal and, hence, perfectly distinguishable. Indeed, the measurement that shall be performed based on Eq. (6) is $\mathbf{A}' = \{P_+, P_-\}$ for the corresponding outcomes $\textcircled{1}$ and $\textcircled{2}$; operators P_{\pm} are projectors into the σ_x eigenvectors, i.e. states $|\pm\rangle = |\pm\rangle$.

This known result was presented in [22]. The computation serves as a formalized way of obtaining the coupling transformation. Comparing the results, one finds that $A = 2\omega$ and the pre-measurement state from [22] equals to state from Eq. (9) up to an unimportant local phase which is due to a slightly different choice of completing unitary V in Eq. (7).

C. State measurement as the first measurement

With the framework presented in previous section we are able to choose also a different measurement as the first one, it does not have to be the conclusiveness measurement. Let us choose state 1 measurement (see Fig. 7b), i.e. we want to know whether the presented state $|\psi\rangle$ is $|\psi_1\rangle$. If we are presented answer $\textcircled{1}$, we know that the given state was $|\psi_1\rangle$; in the opposite case of answer $\textcircled{1}$ we need to perform second measurement that shall tell us whether the state was $|\psi_2\rangle$ or the outcome is inconclusive $\textcircled{?}$.

In the previous case of conclusiveness measurement as the first one to be performed, we saw that in the first step we either got an inconclusive answer, in which case the outcome states for both input states were the same, or we got a conclusive answer, in which case the outcome states for the two input states were orthogonal, and thus perfectly distinguishable.

What can we expect in this scenario, where we first test whether the first state is on the input? Let us assume first, that we got the first state on the input; the test shall then show with probability p_1 that we have state $|\psi_1\rangle$ and with complementary probability we obtain outcome $\textcircled{1}$ and we need to perform the second measurement, where answer that we have state $|\psi_2\rangle$ has zero probability and so outcome $\textcircled{?}$ will be always given. In the case we are presented with state $|\psi_2\rangle$, the first measurement has to always provide outcome $\textcircled{1}$; the subsequent measurement shall afterwards lead to conclusive answer $\textcircled{2}$ with probability p_1 and with probability $p_?$ lead to inconclusive outcome $\textcircled{?}$. Let us confirm these expectations.

The unitaries used in the construction of the coupling in Eq. (8) and the basis transformation U_B are determined to be

$$U_B = |0\rangle\langle\psi_2^\perp| + |1\rangle\langle\psi_2| = \begin{pmatrix} \sin\omega & \cos\omega \\ \cos\omega & -\sin\omega \end{pmatrix},$$

$$V_0 = \frac{1}{\sqrt{2}\cos\omega} \begin{pmatrix} 1 & \sqrt{\cos 2\omega} \\ \sqrt{\cos 2\omega} & -1 \end{pmatrix},$$

$$V_1 = -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The second measurement $A' = \{A', \mathbb{1} - A'\}$ is determined based on Eq. (6). Choosing effect A' to correspond to outcome $\textcircled{2}$, and the complementary effect to outcome $\textcircled{?}$ we obtain

$$A' = \left(P_2 + \frac{1}{\sqrt{1-\lambda}} P_2^\perp \right) \lambda P_1^\perp \left(P_2 + \frac{1}{\sqrt{1-\lambda}} P_2^\perp \right) \quad (10)$$

First, supposing we are given state $|\psi_2\rangle$ on the input, the pre-measurement state is

$$|\psi_2'\rangle = |1\rangle \otimes |1\rangle.$$

The ancillary qubit measurement will measure state $|1\rangle$ with probability 1; this state corresponds to outcome $\textcircled{1}$

and we need to follow with second measurement A' on the post-measurement state $|\tilde{\psi}_2\rangle = |\psi_2\rangle$ — this is the same state as the presented state, as it is the 1-eigenstate of the measurement effect B . The probability that effect A' will be measured in the second measurement, i.e. state $\textcircled{2}$ will be determined, is directly computed using the Born rule

$$p_2 = \text{tr}[P_2 A'] = \lambda \text{tr}[P_1^\perp P_2] = 2 \sin^2 \omega = p_1.$$

Since we always end up doing this measurement, initial state $|\psi_2\rangle$ leads to a definitive answer $\textcircled{2}$ with conclusive probability p_1 and to inconclusive answer $\textcircled{?}$ with probability $p_?$.

Now suppose state $|\psi_1\rangle$ is presented on the input. Particular computations are more extensive than in the previous case, but still straightforward. The pre-measurement state is

$$|\psi_1'\rangle = \sqrt{2} \sin\omega |0\rangle \otimes |0\rangle + \sqrt{\cos 2\omega} |1\rangle \otimes \left(\sqrt{2} \sin\omega |0\rangle + \sqrt{\cos 2\omega} |1\rangle \right).$$

In the case state $|0\rangle$ is measured on the ancillary qubit, outcome $\textcircled{1}$ is assumed and this happens with probability $p_1 = 2 \sin^2 \omega$; the post-measurement state is $|\psi_1\rangle$. With probability $p_? = \cos 2\omega$ outcome $\textcircled{1}$ is provided and we follow with measurement A' . The post-measurement state is

$$|\tilde{\psi}_1\rangle = U_B^\dagger \left(\sqrt{2} \sin\omega |0\rangle + \sqrt{\cos 2\omega} |1\rangle \right)$$

After some computation we find that the probability measurement A' yields outcome $\textcircled{2}$ is $p = 0$ and so, overall, we either obtain outcome $\textcircled{1}$ with probability p_1 or outcome $\textcircled{?}$ with probability $p_?$.

IV. DISCUSSION

We have presented a method of transforming complicated general quantum measurements into a sequence of simple measurements. Similarly to [1] we have provided a framework for the analysis, which we used to study quantum unambiguous state discrimination in its simplest setting. We were able to show that this framework describes exactly the construction of Peres [22] in the case we perform the conclusiveness measurement first.

However, the framework allows us to choose any other measurement as the first one, e.g. the measurement whether we are presented with the first state. In such case we devised the measurement procedure. The reason why one might opt for this option is following. Quantum measurements, even the simple ones, are from the experimental point quite demanding. In current noisy quantum devices this means that during the measurement we lose a lot of quantum resources (coherence in particular). It might be therefore beneficial to perform measurements that give us largest amount of information as early in the process of measurement as possible.

Take for example a case where we are presented with two states that are close to orthogonal. In such case performing measurement for conclusiveness will be followed by state measurement with high probability, but the state presented to this second measurement will be presented with higher noise rate. But if we will perform the measurement for the first state, we will be presented with definite answer with much higher probability and the second measurement will be less frequently performed.

At this point the procedure might not seem very useful, as in this simplest setting the measurement on the original system can be performed irrespective of the measurement on the qubit system. However, in construc-

tions of measurements with larger number of outcomes, this flexibility might become important, as the measurements to be performed will be conditioned on the previous outcomes. The situation becomes apparent already for $n = 4$ outcomes, where subsequent measurements depend on the previous outcome.

We hope that this procedure can offer us with more precise measurement processes on current quantum devices. And while the precise decision on which sequence of partial measurements introduces the least errors depends on many factors, presented framework is independent of the (measurement) implementation and can be used for such tasks.

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Appendix A: Completing qubit coupling unitary

Any qubit effect A can be written in form

$$A = \frac{1}{2}(\alpha \mathbb{1} + \vec{a} \cdot \vec{\sigma}), \quad \mathbb{1} - A = \frac{1}{2}[(2 - \alpha) \mathbb{1} - \vec{a} \cdot \vec{\sigma}].$$

With this parametrization, we can express both $A^{1/2}$ and $(\mathbb{1} - A)^{1/2}$ in their diagonal form suitable for evaluation in V :

$$A^{1/2} = \sqrt{\frac{\alpha + a}{2}} P_{\vec{a}} + \sqrt{\frac{\alpha - a}{2}} P_{-\vec{a}},$$

$$(\mathbb{1} - A)^{1/2} = \sqrt{\frac{2 - \alpha - a}{2}} P_{\vec{a}} + \sqrt{\frac{2 - \alpha + a}{2}} P_{-\vec{a}},$$

where $P_{\pm\vec{a}}$ are projectors into direction of $\pm\vec{a}$. We can write V as

$$V = \begin{pmatrix} \sqrt{\frac{\alpha+a}{2}} & \bullet & 0 & \bullet \\ \sqrt{\frac{2-\alpha-a}{2}} & \bullet & 0 & \bullet \\ 0 & \bullet & \sqrt{\frac{\alpha-a}{2}} & \bullet \\ 0 & \bullet & \sqrt{\frac{2-\alpha+a}{2}} & \bullet \end{pmatrix}$$

with \bullet standing for incomplete elements. There is some freedom in the unitary completion even in the case when we choose to make it block-diagonal, with our choice being

$$V = \begin{pmatrix} \sqrt{\frac{\alpha+a}{2}} & \sqrt{\frac{2-\alpha-a}{2}} & 0 & 0 \\ \sqrt{\frac{2-\alpha-a}{2}} & -\sqrt{\frac{\alpha+a}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\alpha-a}{2}} & \sqrt{\frac{2-\alpha+a}{2}} \\ 0 & 0 & \sqrt{\frac{2-\alpha+a}{2}} & \sqrt{\frac{\alpha-a}{2}} \end{pmatrix}.$$

This block-diagonal matrix can be rewritten as a general control unitary

$$V = |0\rangle\langle 0| \otimes V_0 + |1\rangle\langle 1| \otimes V_1$$

with

$$V_0 = \begin{pmatrix} \sqrt{\frac{\alpha+a}{2}} & \sqrt{\frac{2-\alpha-a}{2}} \\ \sqrt{\frac{2-\alpha-a}{2}} & -\sqrt{\frac{\alpha+a}{2}} \end{pmatrix},$$

$$V_1 = \begin{pmatrix} \sqrt{\frac{\alpha-a}{2}} & \sqrt{\frac{2-\alpha+a}{2}} \\ \sqrt{\frac{2-\alpha+a}{2}} & \sqrt{\frac{\alpha-a}{2}} \end{pmatrix}.$$

Unitary V can be further rewritten as in Eq. (8).