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Topologically Protected Quantization of Work

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The transport of a particle in the presence of a potential that changes periodically in space and in time can be characterized by the amount of work needed to shift a particle by a single spatial period of the potential. In general, this amount of work, when averaged over a single temporal period of the potential, can take any value in a continuous fashion. Here, we present a topological effect inducing the quantization of the average work. We find that this work is equal to the first Chern number calculated in a unit cell of a space-time lattice. Hence, this quantization of the average work is topologically protected. We illustrate this phenomenon with the example of an atom whose center of mass motion is coupled to its internal degrees of freedom by electromagnetic waves.

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Topological phases of matter constitute a new paradigm in condensed matter physics. Remarkable examples are the Haldane anomalous insulator [1], an instance of the more general Chern insulators, and, even more generally, topological insulators and superconductors [2,3]. The later are symmetry protected topological phases of free fermions. Unlike the conventional phases of matter described by the Landau-Ginzburg theory in terms of a local order parameter [4], instead, topological insulators and superconductors are described by topological invariants, such as the holonomy of a flat connection, like the Zak phase [5,6], or the Chern number of a vector bundle [7–10] over a two-torus or an arbitrary Riemann surface, like the Thouless–Kohmoto–Nightingale–den Nijs (TKNN) invariant [11]. These topological invariants measure the nontrivial "twisting" of the wave functions of the bulk, which are usually subject to certain generic symmetries like time reversal, particle-hole, or chiral symmetries. Topological insulators and superconductors were systematically classified, using K theory [12], by Kitaev [13], and using homotopy groups and Anderson localization, by Schnyder, Ryu, Furusaki, and Ludwig [14,15]. The resulting classification exhibits Bott periodicity, twofold for the complex case and eightfold for the real case, and is known as the periodic table of topological insulators and superconductors. Moreover, the bulk-to-boundary principle predicts that, when terminating the system to the vacuum, there will appear gapless modes living in the boundary of the system. These modes are topologically protected. One can understand the existence of these gapless modes by anomaly inflow arguments [16,17].

Topological insulators and superconductors [2,3,18] are very attractive from the experimental point of view due to their robustness to perturbations and also due to the many potential applications to photonics, spintronics, quantum computing, and, more generally, to the emergent field of quantum technologies [19,20].

Experimentally, one can study topological insulators and superconductors in quantum simulators which are versatile systems that can mimic behavior of other systems difficult to control in the laboratory. Among the physical platforms for the quantum simulation of topological matter, ultracold atoms in optical lattices [21,22] and topological photonics [23] offer the most promising realizations. Quantum simulators have allowed for the realization of the topological insulators in one-dimensional (1D) [24–29], 2D [30–32], and even 4D space [33,34]—the latter with the help of a synthetic dimension.

Thirty years ago Thouless proposed the idea of a topological charge pump where transport of charge, described by an adiabatically and periodically evolving Hamiltonian, was quantized and determined by the first Chern number calculated in the time-momentum space [35]. More concretely, if one has a one-dimensional translation invariant gapped system of free fermions on a lattice then, by adiabatically and periodically driving the system, the center of mass position is shifted, in one period of driving, by an integer multiple of the lattice constant. This integer is the

first Chern number of the vector bundle of occupied states in the instantaneous ground states of the system, defined over the space of Bloch momenta and time—topologically, a torus. Direct observation of the Thouless quantum pump was demonstrated in a quantum simulator where bosonic ultracold atoms were prepared in the Mott insulator phase in an optical lattice whose tunneling amplitudes were periodically modulated in time [36,37].

In the present Letter we consider a different phenomenon. Namely, we consider an atom constrained to move in 1D, with internal degrees of freedom subject to a space-time periodic potential coupling the internal states. In this case, it is possible, by preparing the system in a dressed state, that the atom experiences an effective synthetic electric field whose average work, in a period of driving and one wavelength, is quantized in units of the Planck constant h. The quantization is topological in nature and robust against deformations of the system preserving the gap. In the following, we provide an explicit situation where this topological effect occurs and propose a way to experimentally realize it. The differences between this phenomenon and that of Thouless pumping are pointed out in Table I.

Let us consider an atom where the ground state energy level is characterized by the total angular momentum $F = 1$ and in the presence of an external magnetic field the energies E_{m_F}, $m_F = +1, 0, -1$ of the magnetic sublevels are split, $\Delta E = E_1 - E_0 = E_0 - E_{-1}$. We denote the internal states by $|0\rangle \equiv |m_F = 0\rangle$, $|1\rangle \equiv |m_F = +1\rangle$ and $|2\rangle = |m_F = -1\rangle$. If an atom is subjected to two counter-propagating circularly polarized electromagnetic waves of the frequency ω, the internal degrees of freedom of an atom and the electromagnetic fields can be described by the dressed-atom Hamiltonian which, within the rotating wave approximation, reads [21],

$$M(t,x) = \delta(t)(|1\rangle\langle 1| - |2\rangle\langle 2|) + \Omega(t,x)|0\rangle\langle 1| + \Omega^*(t,x)|0\rangle\langle 2| + \text{H.c.}, \tag{1}$$

where we assume that the detuning is oscillating in time due to the periodic modulation of the magnetic field, $\delta(t) = \Delta E(t) - \hbar\omega = \gamma + \nu\cos(\tilde{\omega}t)$, with the frequency $\tilde{\omega} \ll \omega$ and $\nu, \gamma \in \mathbb{R}$. The Rabi frequency depends periodically on time and space, $\Omega(t,x) = \alpha_1(t)e^{ikx} + \alpha_2(t)e^{-ikx}$, where k denotes the wave number of the electromagnetic waves while $\alpha_1(t) = (\alpha/2)\cos(\tilde{\omega}t - \pi/4)$ and $\alpha_2(t) = (\alpha/2)\cos(\tilde{\omega}t + \pi/4)$ describe periodic modulations of the amplitudes of the waves, with the same frequency as the frequency of the magnetic field modulation, where α is proportional to the dipole matrix element. The Hamiltonian $M(t,x)$ is periodic both in space and in time with the periods $\lambda = 2\pi/k$ and $T = 2\pi/\tilde{\omega}$, respectively, and can be written in a more compact form, $M(t,x) = \sum_{\mu=1}^{3} B^\mu(t,x)J_\mu$ where $J_3 = |1\rangle\langle 1| - |2\rangle\langle 2|$, $J_1 - iJ_2 = \sqrt{2}(|0\rangle\langle 1| + |2\rangle\langle 0|)$ and

$$\begin{aligned} B^1(t,x) &= \alpha\cos(kx)\cos(\tilde{\omega}t), \\ B^2(t,x) &= \alpha\sin(kx)\sin(\tilde{\omega}t), \\ B^3(t,x) &= \gamma + \nu\cos(\tilde{\omega}t). \end{aligned} \tag{2}$$

When the atomic center of mass motion is coupled to its internal degrees of freedom certain geometric gauge fields arise [21,38–41]. For simplicity, let us consider that the atomic motion is restricted to one spatial dimension. The full Hamiltonian of the system is given by

$$H = \frac{p^2}{2m} + M(t,x), \tag{3}$$

where x and p are the atomic center of mass coordinate and momentum, m is the mass. We can solve the eigenvalue problem for $M(t,x)$, yielding the eigenvalues $\varepsilon_1(t,x) = |\mathbf{B}(t,x)| = \sqrt{\sum_\mu (B^\mu)^2}$, $\varepsilon_2(t,x) = 0$, and $\varepsilon_3(t,x) = -\varepsilon_1(t,x)$ and the corresponding eigenstates (dressed states of an atom) $|\eta_i(t,x)\rangle$. The most general solution of the Schrödinger equation will be given by a linear combination

TABLE I. Comparison between charge pumping and quantization of work. The σ_μ's denote the usual Pauli matrices.

	Top. effect	
Property	Work quantization	Thouless pumping
Parameter space topologically a torus T^2	Space-time	Bloch momenta and time
Gapped Hamiltonian	$M(t,x) = \sum_{\mu=1}^{3} B^\mu(t,x)J_\mu$	$H(t,k) = \sum_{\mu=1}^{3} d^\mu(t,k)\sigma_\mu$ (in the simplest scenario of a two-band system)
Wave functions (sections)	Dressed states $\lvert\eta(t,x)\rangle$	Bloch states $\lvert u(t,k)\rangle$
Gauge field	$-iA(t,x)/\hbar = \langle\eta(t,x)\lvert d\rvert\eta(t,x)\rangle$ acts as an effective external field	$A(t,k) = \langle u(t,k)\lvert d\rvert u(t,k)\rangle$ manifests through coupling to an external field
Field strength	$F(t,x) = dA$	$F(t,k) = dA$
First Chern number	Average work performed by $E(t,x)$ on the unit cell of space-time lattice	Shift of the center of mass of a system $x_{\text{cm}} = \langle x\rangle$ in one period of driving

$\psi(t,x) = \sum_{i=1}^{3} \Psi^i(t,x)|\eta_i(t,x)\rangle$. Writing the vector $\Psi = (\Psi^1, \Psi^2, \Psi^3)^T$, we get the time-dependent Schrödinger equation corresponding to the Hamiltonian (3) in the form

$$\left[i\hbar\left(\frac{\partial}{\partial t}+\mathcal{A}_0\right)+\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}+\mathcal{A}_1\right)^2-V-\mathcal{E}\right]\Psi=0, \quad (4)$$

where $\mathcal{E} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and the matrices $\mathcal{A}_0 = (\langle\eta_i|\partial_t|\eta_j\rangle)$ and $\mathcal{A}_1 = (\langle\eta_i|\partial_x|\eta_j\rangle)$ are the components of the matrix-valued one-form $\mathcal{A} = (\langle\eta_i|d|\eta_j\rangle)$.

For the configuration of the electromagnetic waves and the detuning we have chosen, the eigenvalues $\varepsilon_i(t,x)$ are separated from each other by gaps for each (t,x). If we prepare an atom in, e.g., the positive energy dressed state,

$$|\eta_1(t,x)\rangle = \frac{|1\rangle + \sqrt{2}z|0\rangle + z^2|2\rangle}{1+|z|^2}, \qquad z(t,x) = \frac{B^1 + iB^2}{|\mathbf{B}| + B^3}, \quad (5)$$

it will follow this state in the time evolution provided its kinetic energy is much smaller than the gap between the adjacent dressed state levels. Then, one can perform an adiabatic Born-Oppenheimer approximation and project the dynamics onto this state only [21,39–41]. The resulting effective Schrödinger equation for the center of mass wave function, $\phi(t,x) = \Psi^1(t,x)$, is that of a particle in the presence of an external gauge field, $A_0/\hbar = i\langle\eta_1|\partial_t|\eta_1\rangle$ and $A_1/\hbar = i\langle\eta_1|\partial_x|\eta_1\rangle$, and an effective scalar potential V_{eff},

$$i\hbar\frac{\partial\phi}{\partial t} = \left(\frac{1}{2m}(p-A_1)^2 - A_0 + V_{\mathrm{eff}}\right)\phi, \quad (6)$$

with

$$V_{\mathrm{eff}} = \frac{\hbar^2}{2m}g_{11}(t,x) + \varepsilon_1(t,x), \quad (7)$$

where $g_{11} = \sum_{j>1} |\langle\eta_1|(\partial M/\partial x)|\eta_j\rangle|^2/(\varepsilon_j - \varepsilon_1)^2$ is the 11th component of the *quantum metric* [41–43].

Since the atom is constrained to move in a single space dimension, the only relevant component of the field strength tensor is the synthetic electric field force acting on a particle of a unit charge

$$E(t,x) = \frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x} = \hbar\frac{\mathbf{B}\cdot\left(\frac{\partial\mathbf{B}}{\partial x}\times\frac{\partial\mathbf{B}}{\partial t}\right)}{|\mathbf{B}|^3}. \quad (8)$$

Because $\mathbf{B} \equiv (B^1, B^2, B^3)$ is space-time periodic, one can define a Chern number c_1 associated to the positive energy dressed state which will be minus twice the winding number of the map $(t,x) \mapsto \mathbf{B}(t,x)/|\mathbf{B}(t,x)| \in S^2$, where S^2 denotes the unit sphere in $\mathbb{R}^3$, i.e., $c_1 = (1/2\pi\hbar)\int_0^T\int_0^\lambda E(t,x)dtdx$. In particular, with $\alpha/\nu = 1$, using Eq. (2), we get a nontrivial Chern number -4 for $-1 < \gamma/\nu < 0$ and 4 for $0 < \gamma/\nu < 1$ and trivial elsewhere [44]. A similar result holds for the negative energy dressed state but with the Chern number being the opposite. The zero energy dressed state always has a trivial Chern number.

The quantization of the Chern number, proved in [44], amounts to having, on the unit cell of the space-time lattice, a quantized value for the flux $\int E(t,x)dtdx$ in units of $2\pi\hbar \equiv h$. Now $E(t,x)dx$ is, dimensionally, the amount of work, of the *electric field force*, under the displacement dx of a particle with a unit charge. The space-time lattice involved is simply $\Lambda = \{(t,x) = (mT, n\lambda), m, n \in \mathbb{Z}\}$. The interpretation of the quantized value of the Chern number is the following: the average over a period T of the work performed by the electric field E in the transport of a classical particle by a distance of a single space cell, i.e., $x \to x + \lambda$, is quantized in units of Planck's constant h. If we consider the normalized average in time, to have proper units of work, we get $(1/T)\int_0^T\int_0^\lambda E(t,x)dtdx = (h/T)c_1 = (\hbar\tilde{\omega})c_1$, with $c_1 \in \mathbb{Z}$ the Chern number. We thus get quantization in units of the driving energy $\hbar\tilde{\omega}$.

A possible way to experimentally observe quantization of the average work, in the example we consider, can be done indirectly as follows. Take the time interval $[0,T]$ and consider a number N of instants t_i, $i = 1, \ldots, N$. For each instant t_i, prepare an atom in the dressed state band with energy $\varepsilon_1(t_i,x)$ and described, at (t_i,x), by $|\eta_1(t_i,x)\rangle$. We want the state of the center of mass degree of freedom of an atom to be strongly localized in a certain point $x(t_i)$ (i.e., much better than the size of a single space cell which is not a problem if the experiment is performed in the rf range where the wavelength λ is of the order of the meter), so the dynamics for time $t \in [t_i, T]$ is well described by the classical equations of motion

$$m\frac{d^2x}{dt^2}(t) = E(t,x(t)) - \frac{\partial\varepsilon_1}{\partial x}(t,x(t)) - \frac{\hbar^2}{2m}\frac{\partial g_{11}}{\partial x}(t,x(t)). \quad (9)$$

We then measure the position of the atoms in the period $[0,T]$. With the resulting trajectories $x_i(t)$, $t \in [t_i, T]$, $i = 1, \ldots, N$, we can then differentiate with respect to time twice obtaining the acceleration. With this procedure we will get a profile of the total force field in the unit cell of Λ which we can compare to the theoretical predictions. Because of the localization of the center of mass of an atom, the observed profile should be the same and quantization of the time average of work of $E(t,x)$ can be confirmed. In Fig. 1 we show how the sampling of the accelerations of trajectories allows us to have the force profile on the unit cell. Additionally, we show the profile of $E(t,x)$ on the unit cell. The total force and the profile of $E(t,x)$ are qualitatively similar. The reason is that all

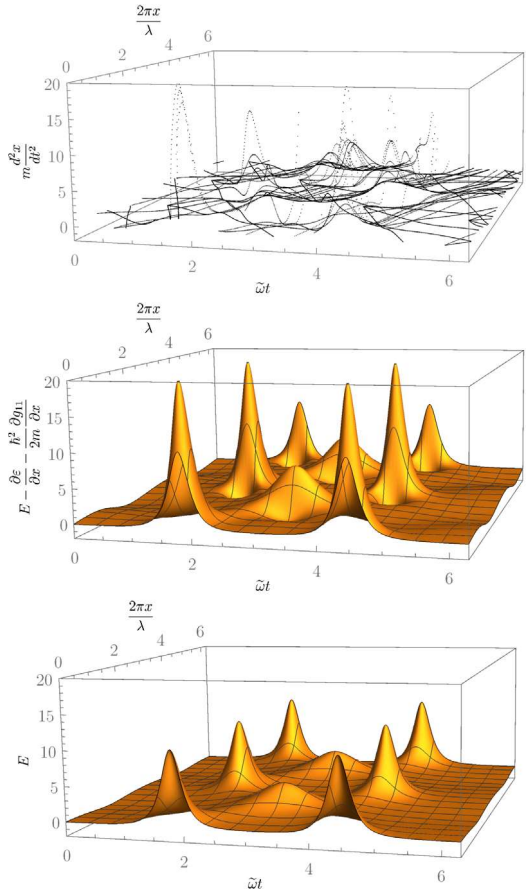

FIG. 1. Proposal for experimental demonstration of the quantization of the average work performed by the electric field $E(t,x)$. The top panel shows acceleration profile in a space-time unit cell obtained by integration of the classical equations of motion (9). For each initial condition, the resulting classical trajectory allows one to calculate the acceleration by differentiation of the trajectory with respect to time twice. In the middle panel we show the same profile obtained by plotting the total force acting on an atom. In the bottom panel the contribution from the electric field force $E(t,x)$ is plotted only. The integration of $E(t,x)$ over a single space-time cell results in a quantized value which can be also obtained when one estimates the integral with the help of the points presented in the top panel. In the latter case, one has to first subtract the other contributions to the total force which are known from the theoretical description, cf. Eq. (9). Here, $\mu = 0.5$ and $m = 1$. The Chern number is 4. Moreover, we take $\hbar = 1$.

contributions to the total force increase with the decrease of the gap function $\varepsilon_1(t,x)$.

We would like to remark that the phenomenon we describe is at the boundary between classical and quantum physics. Classical, since we want the states of the atoms to be strongly localized, so that the dynamics is classical. This is achieved by staying in the cold atom regime and not in the ultracold one. Quantum, since the atom will experience the effect of a *synthetic force* whose average work on the unit cell is quantized due to the quantum nature of the wave function of the internal degrees of freedom of the atom. This is achieved by having a gap which is larger than the kinetic energy of the atom.

We stress that the topological effect of work quantization considered in this Letter and that of Thouless pumping are physically different, although mathematically similar, cf. Table I. Explicitly, in our case, it is the topology of quantum states over space-time and not of quantum states over Bloch momentum space that is involved. This topology is then reflected in the *quantization* of the average work of the synthetic electric field and not the quantization of the shift of the center of mass of the system.

Finally, we would like to make contact with the very recent Ref. [45] where a topological energy pump in 1D, in the context of a driven system was considered. There, a "work polarization" is quantized. We remark that the topological invariant there refers to the homotopy class of a map $\mathcal{P}$, describing the dynamics within each cycle, from a three-dimensional torus to the unitary group U(2). The three-dimensional torus is parametrized by variables (t,λ,k) where t is time, λ is a flux, and k is the one-dimensional momentum in the first Brillouin zone. As a consequence, although in both cases there is quantization of some type of work, just as the Thouless pumping is significantly different from the phenomenon considered here, see Table I, so is this one.

In summary, we have presented an effect in which the transport of a particle in the presence of a space-time periodic potential is characterized by a quantized average, over a period of the potential, amount of work needed to shift a particle by a single spatial period of the potential. The quantization was understood in terms of the topological twist of the vector bundle of dressed states. Moreover, we have provided an experimental procedure to probe this phenomenon.

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[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.020601 for a detailed description of the topological properties of the system considered here and the derivation of the associated Chern numbers.

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